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LETTER TO THE EDITOR

Three-dimensional turbulent diffusion versus critical phenomena

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Abstract. The critical exponent, ν, of the correlated (or coherent) length of a turbulent cluster has been related to the minimal fractal dimension, D_{min}, of multifractal isotropic turbulence. The relationship has turned out to be $\nu D_{min} = 3/2$. For the homogeneous case, $D_{min} = 3$ and hence, $\nu = 1/2$ (the so-called mean-field approach of the theory of critical phenomena). For $D_{min} \simeq 2.36$ (the well known turbulent value), $\nu \simeq 0.63$. This result allows one to classify this case as the critical phenomenon of the so-called 'thermal' class of universality. Transition to the 'percolation' class of universality ($\nu \simeq 0.9$) is determined by the boundary conditions.

The process of formation of turbulent clusters is known to be a critical phenomenon (see, for example, [1–3]). Indeed, suppose that initially, there exist only the large-scale (l_0) fluctuations of a passive scalar. The cascade process results in the formation of hierarchy of fluctuations with scales $l_n \propto q^{-n} l_0$, where q is the multiplicity of the breakdown process of larger scales into smaller ones. The characteristic time of the nth order of breakdown process (for large enough n) can be estimated as

$$T_c^{(n)} \propto \left\{ \frac{\langle u^2 \rangle}{\langle c^2 \rangle} k_n^3 E_c(k_n) \right\}^{-1/2} \tag{1}$$

where $k_n \propto l_n^{-1}$ is wave number, $\langle u^2 \rangle$ is the mean energy of turbulent velocity fluctuations, $\langle c^2 \rangle$ is the mean square of the concentration fluctuations of the passive scalar, and $E_c(k)$ is the spectral density of the concentration fluctuations.

The total time of the cascade formation is

$$t_\infty \propto \sum_{n=0}^{\infty} T_c^{(n)}. \tag{2}$$

The convergence of the series in (2) depends on the behaviour of $T_c^{(n)}$ at large n. If at large k

$$E_c(k) \propto k^{-\gamma} \tag{3}$$

it is easy to show that t_∞ is *finite* if $\gamma < 3$.

In the following we will be interested in the quantity

$$t_\infty - t_N \propto \sum_{n=N}^{\infty} T_c^{(n)} \tag{4}$$

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for sufficiently large N, so that we can use the estimate (3). It follows from (3) and (4) that

$$t_\infty - t_N \propto \sum_{n=N}^{\infty} l_n^{(3-\gamma)/2} \propto \sum_{n=N}^{\infty} q^{-n(3-\gamma)/2} \propto q^{-N(3-\gamma)/2}. \tag{5}$$

If after N breaking ups there will form $M(t_N) \propto q^N$ fluctuations at scale $l_N \propto q^{-N} l_0$ from an initial fluctuation of scale l_0, then the set of the fluctuations (a cluster) will occupy some volume with effective scale l_*, and

$$M(t_N) \propto l_*^D \tag{6}$$

where D is the fractal dimension of the formed turbulent (passive scalar) cluster.

Generally, turbulence is a multifractal phenomenon (see, for example, [4]). If we want to interpret l_* as the upper bound of the scaling range where the passive scalar clusters behave self-similarly (and hence may be characterized by a fractal dimension), then we should put $D = D_{\min}$ in (6) [5]. In this case, l_* can be estimated as so-called correlation (or coherence) length.

Taking into account that (from (5))

$$M(t_N) \propto (t_\infty - t_N)^{2/(\gamma-3)}$$

it follows (from (6))

$$l_*(t_N) \propto (t_\infty - t_N)^{-2/D(3-\gamma)} \tag{7}$$

i.e., if $\gamma < 3$, (7) is a singular relation with $l_* \to \infty$ as $t_N \to t_\infty$. The critical exponent ($l_*(t) \propto (t_\infty - t_N)^{-\nu}$):

$$\nu = \frac{3}{(3-\gamma)D_{\min}}. \tag{8}$$

If we are interested in the Kolmogorov turbulence (with Corrsin–Obuchov spectral law: $\gamma = 5/3$) [6], then

$$\nu = \frac{3}{2D_{\min}}. \tag{9}$$

In the homogeneous case, $D_{\min} = 3$, and hence,

$$\nu = \frac{1}{2}. \tag{10}$$

This value of the critical exponent is known in the theory of critical phenomena as so-called classical (or mean-field) value. However, in reality $D_{\min} < 3$ [4,5]. If $D_{\min} = 7/3$ [4,5], then (from (9))

$$\nu = 9/14 \simeq 0.64. \tag{11}$$

Turbulent experiments and multifractal considerations give $D_{\min} \simeq 2.36$. Then (from (9))

$$\nu \simeq 0.63. \tag{12}$$

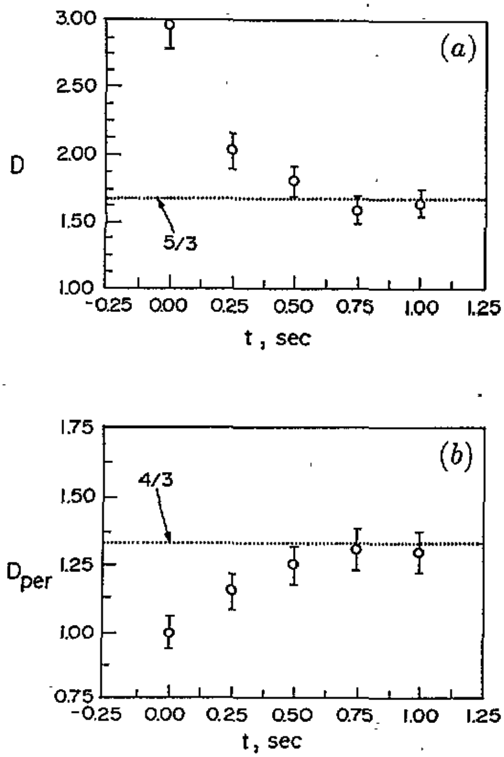

Figure 1. The evolution of the fractal dimension of the passive scalar cloud (*a*), and its perimeter (*b*) [9].

This value of critical exponent ν is precisely the same as the value of ν in the so-called 'thermal' class of universality of the theory of critical phenomena (see, for example, [8]).

Results (11) and (12) have been obtained for $D_{\rm min} \simeq D_\sigma$, where D_σ is the fractal dimension of external surface of a fractal cluster [5]. However, there is a possibility for $D_{\rm min} < D_\sigma$. The fractal dimension of intersection of the external surface and the inner part of the cluster, $D_\cap$, should be a topological one in this case, i.e.,

$$D_\cap = \min[D_\sigma \text{ or } D_{\rm min}]$$

where [...] means the integer part. On the other hand,

$$D_\cap = D_\sigma + D_{\rm min} - d$$

where d is the topological dimension of the space. Hence,

$$\min[D_\sigma \text{ or } D_{\rm min}] = D_\sigma + D_{\rm min} - d. \tag{13}$$

For the theoretical value $D_\sigma = 7/3$ [4, 5] it follows from (13) that $D_{\rm min} = 5/3$. The results of recent numerical simulation of turbulent dispersion [9] are shown in figure 1 (adapted from [9]). The dispersion process in this simulation was realized by releasing a cloud of tracer particles into a three-dimensional velocity field composed of a linear superposition of so-called Taylor–Green vortices. The wave numbers and amplitudes of these vortices were prescribed by Weierstrass functions. Such a velocity field is solenoidal and has a Kolmogorov intertial range power law spectrum. It is seen in figure 1(a) that fractal dimension of the cloud of tracer particles (D) reduces from 3 at the initial moment to about 5/3 at $t = 1$ second. At the same time the dimension of the perimeter of the projection of the passive scalar cloud external surface increases from 1 ($D_\sigma = 2$) at $t = 0$ to about 4/3 ($D_\sigma \simeq 7/3$) at $t > 0.5$ second (figure 1(b)).

The results are in good agreement with (13). If we take the value of $D_{\rm min} \simeq 5/3$, then from (9) we obtain

$$\nu \simeq 0.9. \tag{14}$$

This value of the critical exponent ν is the same as the value of ν in the so-called 'percolation' class of universality of the critical phenomena (see, for example, [10, 1, 2]).

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